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### A Strategic Market Game with a Mutual Bank with Fractional Reserves and Redemption in Gold (A Continuum of Traders)

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AT YALE UNIVERSITY

Box 2125, Yale University  
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COWLES FOUNDATION DISCUSSION PAPER NO. 964

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A STRATEGIC MARKET GAME WITH A MUTUAL BANK  
WITH FRACTIONAL RESERVES AND REDEMPTION IN GOLD  
(A CONTINUUM OF TRADERS)

by

M. Shubik and D.P. Tsomocos

December 1990

# A STRATEGIC MARKET GAME WITH A MUTUAL BANK WITH FRACTIONAL RESERVES AND REDEMPTION IN GOLD (A CONTINUUM OF TRADERS)

M. Shubik and D. P. Tsomocos \*

**ABSTRACT** We utilize the strategic market game approach to analyze the role and function of a mutual bank with variable fractional reserves, redemption in gold and endogenous interest rate formation. We specify the conditions of enough money and its distribution. Using the continuum of traders model, we show existence and optimality for the case of *no* bankruptcy as well as for the case in which there exists the potentiality of bankruptcy. Finally, we analyze the relationship of the gearing ratio and the bankruptcy penalty with respect to the resulting equilibrium allocations.

## 1. THE FINANCING OF TRADE

On these principles it will be seen that it is not necessary that paper money should be payable in specie to secure its value; it is only necessary that its quantity should be regulated according to the value of the metal which is declared to be standard.

Ricardo,<sup>1</sup> p. 239

Experience, however shows that neither a state nor a bank ever have had the unrestricted power of issuing paper money without abusing that power; in all states therefore, the issue of paper money ought to be under some check and control; and none seems so proper for that purpose as that of subjecting the issuers of paper money to the obligation of paying their notes either in gold coin or bullion.

Ricardo, p. 241

In a one period economy the only need for a money or means of exchange is to finance the mechanism of short term trade. In two previous papers, examining exchange with a single commodity money (Shubik, 1990) it was observed that three conditions prevail concerning the amount of gold or other commodity money used as a means of exchange. They are: (1) Enough money, well distributed; (2) enough money poorly distributed and (3) not enough money.

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\*The authors wish to acknowledge the help of discussions held with Pradeep Dubey, John Geanakoplos and Jingang Zhao.

<sup>1</sup>D. Ricardo, 1965, *The Principles of Political Economy and Taxation* (London: J. M. Dent & Sons), based on 3rd ed., 1821.

### Enough Money, Well Distributed

The game  $\Gamma$  will have enough money that is well distributed with respect to a CE  $k$  if:

$$\sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] \leq a_{m+1}^i$$

for all  $i$ , where  $p_{jk}$  are the prices at the  $k^{\text{th}}$  CE and  $x_{jk}^i$  is the final holding of good  $j$  by individual  $i$  at CE  $k$ .<sup>2</sup> There are  $m+1$  goods and the  $m+1^{\text{st}}$  is the money.

### Enough Money, Badly Distributed

The game  $\Gamma$  will have enough money that is badly distributed with respect to a CE  $k$  if:

$$\sum_{i=1}^n \sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] \geq \sum_{i=1}^n a_{m+1}^i$$

and for some  $i$

$$\sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] > a_{m+1}^i .$$

### Not Enough Money

The game  $\Gamma$  will not have enough money with respect to a CE  $k$  if:

$$\sum_{i=1}^n \sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] > \sum_{i=1}^n a_{m+1}^i .$$

## 1.1. From Gold to Fractional Reserves

In the previous papers noted a money market is introduced where individuals can lend gold, but all payments are still made in gold. When there is an absolute shortage of gold needed for transactions a positive rate of interest appears and trade is not Pareto optimal.

In this paper a way to overcome the shortage of gold is considered by introducing a mutual bank with paper currency redeemable in gold. The amount of paper issued is a multiple of the gold held.

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<sup>2</sup>In the game price is given by  $\Sigma b_j^i / \Sigma q_j^i$ . It is also important to note that the inequalities are specified for the instance where the individuals are assumed to either buy or sell; buying *and* selling in the same market would call for more money.

## 2. A MUTUAL BANK WITH REDEMPTION IN GOLD

In constructing a playable exchange game making use of a mutual bank the granting of credit and the accumulation of debt can be centralized. Instead of having to be concerned with individual indebtedness between  $i$  and  $j$ , the bank is a full intermediary. If  $i$  has a line of credit from the bank and pays  $j$ ,  $i$ 's default is against the bank, not  $j$ . The financial damage to  $j$  by  $i$ 's default may come in an aggregated form through  $j$ 's part ownership of the bank.

A mutual bank exists to facilitate trade and thus is not necessarily a profit maximizer. As it is a mutual bank, if it incurs losses these may be prorated among the depositors.

Figure 1 shows the extensive form of the game. At the first move individuals may deposit gold in the mutual bank. The bank ownership is assigned in proportion to the amount deposited. The amount of gold deposited is multiplied by a gearing ratio (given as a parameter of the system) and this determines the total amount of credit offered to all borrowers. At the second move all who wish to borrow bid for bank money. At the third move they exchange goods for bank money. We avoid the informational complexities which may arise when traders have perfect information by assuming a continuum of traders, simultaneous moves at all stages and a minimum of information.

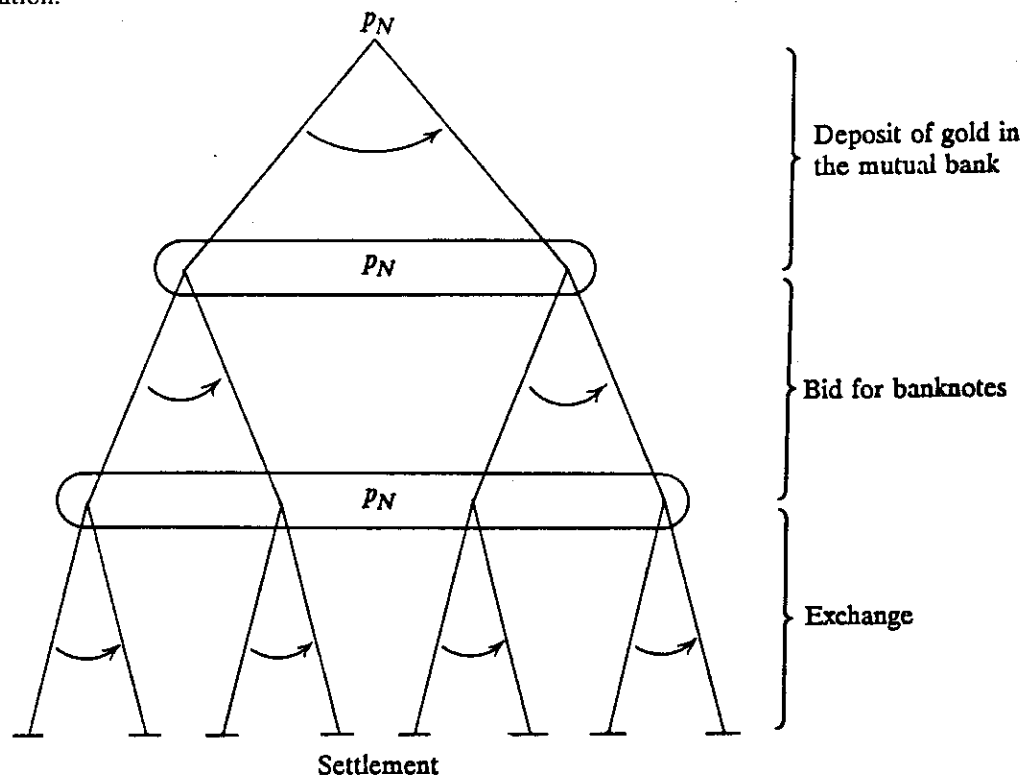


Figure 1

Trade with a mutual bank

## 2.2. Description of the Model

There is a continuum of traders  $T = [0, n]$  which consists of  $n$  types of traders,  $T_i = [i-1, i)$  where  $a^i = a^i$ ,  $\phi^i = \phi^i$  for  $t \in T_i$ . The endowments of traders are  $a^i = (a_1^i, \dots, a_m^i, a_{m+1}^i)$  where  $a^i \in \mathbb{R}_+^{m+1}$ . For  $1 \leq i \leq n$  the utility functions are,  $\phi^i : \mathbb{R}_+^{m+1} \rightarrow \mathbb{R}$ . Note that when we introduce the potentiality of bankruptcy we need to extend the domain by  $\mathbb{R}_+$ . We assume

- (i)  $\sum_{i=1}^n a^i > 0$
- (ii)  $a^i \neq 0$ , for all  $i = 1, \dots, n$
- (iii)  $\phi^i$  is continuous, concave and strongly monotonic for all  $i = 1, \dots, n$ .

The strategy set of player  $i$  is of the form,

$$(1) \quad S^i = \{u^i, v^i, b_1^i, q_1^i, \dots, b_m^i, q_m^i, b_{m+1}^i, q_{m+1}^i\}$$

where the arguments appearing in  $S^i$  are constrained as follows,

$$(2) \quad \left\{ \begin{array}{l} (a) \quad 0 \leq u^i \leq a_{m+1}^i \\ (b) \quad 0 \leq v^i \leq k \sum_{i=1}^n a_{m+1}^i, k \geq 1 \\ (c) \quad \sum_{j=1}^{m+1} b_j^i \leq 1, \text{ for all } j = 1, \dots, m \\ (d) \quad q_j^i \leq a_j^i \text{ for all } j = 1, \dots, m \\ (e) \quad q_{m+1}^i + u^i \leq a_{m+1}^i \end{array} \right.$$

with

$u^i$  = amount of gold deposited in the bank

$k$  = the gearing ratio for bank note issue

$v^i$  = the amount of I.O.U. notes bid by  $i$  for banknotes

$q_j^i$  = amount of good  $j$  offered for sale

$b_j^i$  = percentage of gold bid on good  $j$

It should be noted that one may naturally expect that traders would not offer to repay more money than the total money supply in the economy. Also the presence of a default or bankruptcy penalty provides a motivation against unreasonably high bidding, but the physical bound is sufficient.

A trader type  $i$ 's share of ownership of the mutual bank is specified by:

$$(3) \quad \theta^i = \frac{u^i}{\int_T u^i}.$$

At the second stage of the game the bank, which is a strategic dummy (that is why we suppress its move in the extensive form of Figure 1) offers to issue banknotes up to the level of  $M = k\bar{u}$ , where  $\bar{u} = \sum_{i=1}^n u^i$  and  $k$  is exogenously set.

An endogenous interest rate is formed and is given by:

$$(4) \quad 1+\rho = \frac{\int_T v^i}{k \int_T u^i} = \frac{\int_T v^i}{M}.$$

The last stage of the game  $\Gamma$  consists of the offer-for-sale model as in Dubey-Shubik [3]. In order to study the simplest set of strategies we may imagine that all agents are completely uninformed at each stage, thus strategies remain a set of numbers rather than complex functions of information. This can be done by having players name ratios for the amount that they intend to spend in different markets as in 2(c).

The actual amount of banknotes in possession of  $i$  offer the second move is given by:

$$(5) \quad w^i = \frac{v^i M}{\int_T v^i}.$$

Using the restrictions (2) on  $S^i$ , we can specify the strategy set as,

$$(6) \quad \Sigma^i = \{s \in S^i \text{ subject to (2)}\}.$$

It is easily seen that  $\Sigma^i$  is compact and convex. We will employ the Dubey-Shubik offer-for-sale price formation mechanism. Thus,

$$(7) \quad p_j = \begin{cases} \frac{\int_T b_j^i w^i}{\int_T q_j^i}, & \text{if } \int_T b_j^i w^i > 0 \text{ and } \int_T q_j^i > 0 \\ 0, & \text{otherwise} \end{cases}.$$

The final allocations of the first  $m$  goods is given by:

$$(8) \quad x_j^i = a_j^i - q_j^i + \frac{b_j^i w^i}{p_j} \quad \forall j = 1, \dots, m.$$

The final amount of gold which also indicates the trader's debt position at the end of exchange is specified by:

$$(9) \quad D^i = \left\{ a_{m+1}^i - u^i \right\} + \left\{ \frac{v^i}{1+\rho} - \phi^i - p_{m+1} g_L^i \right\} + \left\{ \sum_{j=1}^{m+1} p_j q_j^i - \sum_{j=1}^{m+1} b_j^i w^i \right\} \\ + \theta^i \left\{ M - \frac{\sum_{i=1}^n v^i}{1+\rho^*} + \sum_{i=1}^n \phi^i + p_{m+1} \sum_{i=1}^n g_L^i \right\}$$

where

$\hat{v}^i$  = size of repayment of his loan in banknotes

$g_L^i$  = size of repayment of his loan in gold

$p_{m+1} g_L^i$  = par value of gold used for repayment

$\rho$  = *ex ante* interest rate

$\rho^*$  = *ex post* interest rate

The first two terms are the amount of gold retained. The next three are the size of loan and the size of repayment. The next two terms are the balance from buying and selling commodities. Finally, the last terms represent the liquidation of the mutual bank, the payment of profits and the return of the original deposit to a depositor in the mutual bank.

Since we allow repayment of loan either using remaining banknotes after exchange or by gold at one-to-one basis (i.e.  $p_{m+1} = 1$ ), we can potentially encounter six different cases depending on whether a trader bankrupts or not. In Appendix 1, we present the cases of the general formulation of (9) and present an explicit calculation for the case where there is no bankruptcy and all loans are paid back only by banknotes. It is important to note that the specification of repayment of loan either by banknotes or by gold at one-to-one basis is what bounds the interest rate from above and allows us to show optimality.

Several items must be noted. If we restrict the use of gold so that it is not a direct means of payment, but can be used only as bank reserves or for consumption purposes then we can have a market for gold where it is bought for banknotes if gold can be used for payments in trade then the money supply in the hands of the public includes both the banknotes issued against the gold deposited and the gold held by the public.

The interest rate formed in the second stage and denoted by  $\rho$  is the contractual (or *ex ante*) rate which is used to determine how much any borrower has agreed to pay. It may be that a borrower is unable to meet his obligation and defaults. In this case the depositors of the mutual will receive less than promised from the mutual. There will be an effective rate  $\rho^*$  where  $\rho^* < \rho$  which is the *ex post* rate of interest taking into account the default of various traders.



The calculation of  $\rho^*$  is complicated by a possible "domino effect" in default among traders who are both depositors and debtors. If individual  $i$  goes into default he lowers the dividend to be received by  $j$ , but this could cause  $j$  to default. In Appendices 2 and 3 we provide flow diagrams for the calculation of the *ex post* interest rate for both the case of finite number of traders and the continuum case we analyze in this paper.

The final payoff to trader type  $i$  is given by:

$$(10) \quad \Pi^i(s) = \phi^i(x^i(s)) - \mu D_+^i(s)$$

where  $D_+^i(s) = \max[0, D_i(s)]$  and  $\mu$  is the bankruptcy penalty in case of default.

In the next section we will show existence and optimality for the case where there is no bankruptcy (i.e.  $D^i \leq 0$ )<sup>3</sup> and for the case where the potentiality of bankruptcy is present. Before proving these results we state some definitions.

An allocation  $(x, p)$  of an exchange economy  $E(n, \phi^i, a^i)$  where  $x^i \in \mathbb{R}_+^{m+1}$ ,  $p \in \mathbb{R}_{++}^{m+1}$  is a *Walras Equilibrium* (W.E.) if

$$(i) \quad \sum_{i=1}^n x^i = \sum_{i=1}^n a^i$$

and

$$(ii) \quad x^i = \operatorname{argmax}\{\phi^i(y) : y \in \mathbb{R}_+^{m+1}, py = pa^i\} \quad \forall i = 1, \dots, n.$$

A *Nash Equilibrium* (N.E.) of  $\Gamma(n, \phi^i, a^i, s^i)$  is an integrable  $s$  such that  $\forall t \in T$  and  $a^t \in S = \prod_{i=1}^n S^i$ ,

$$\Pi^t(s/\alpha^t) \leq \Pi^t(s)$$

where  $(s/\alpha^t)$  is  $s$  with  $s^t$  replaced by  $\alpha^t$ .

If  $s^t = s^{t'}$  for every  $t, t' \in T_i$  and  $\forall i = 1, \dots, n$ ,  $s$  is a type symmetric play and that N.E. is *Type-Symmetric* (T.S.N.E.).

An *Active T.S.N.E.* of  $\Gamma(n, \phi^i, a^i, s^i)$  is a T.S.N.E. with  $\rho > -1$  and  $p_j > 0 \quad \forall j = 1, \dots, m+1$ .

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<sup>3</sup>Bankruptcy can be avoided by selecting a sufficiently harsh penalty so that nobody will be induced to strategically bankrupt. This can be achieved by setting  $\mu = \max(\lambda_1, \dots, \lambda_n)$  where the  $\lambda$ 's reflect the marginal utility of an extra unit of gold at equilibrium. For more on this see Shubik [8] and Shubik-Wilson [10].

## 2.2. Existence and Optimality<sup>4</sup>

**THEOREM 1.** *The strategic market game  $\Gamma(n, \phi^i, a^i, s^i)$  with a continuum of traders of  $n$  types and a mutual bank for any non-zero amount of gold and any gearing ratio  $k \geq 1$  and a sufficiently harsh bankruptcy penalty  $\mu$  which results into no bankruptcy has an active T.S.N.E.*

### PROOF

STEP 0.

(i) Let an outside agency place a fixed bid and a fixed offer  $\epsilon > 0$  in each trading post as well as in the mutual bank. So, let the modified strategy space be,

$$(\Sigma^i)^\epsilon = \{s \in \Sigma^i : s \geq \epsilon, \forall s \in \Sigma^i\}$$

Note that we do not need to place upper bounds because by construction we have naturally imposed bounds on the arguments of the strategy sets.

(ii)  $\Gamma^\epsilon$  is the  $\epsilon$ -modified game where we restricted the strategies of the players to  $(\Sigma^i)^\epsilon$ . Then for a  $n$ -tuple  $s$ , the map from strategies to outcomes,  $(s/\alpha^t) \rightarrow (x^t, D^t)$  where  $D^t = 0$  since utilities are locally non-satiated, is a linear function where  $\alpha^t \in (\Sigma^i)^\epsilon$  since  $\alpha^t$  cannot affect either  $p$  or  $\rho$  in the continuum. Thus  $\Pi^t(s/\alpha^t) = \phi^t(x^t(s))$  is continuous and concave. Also,

$$(BR^i)^\epsilon : \Sigma^\epsilon \rightarrow (\Sigma^i)^\epsilon \text{ is convex,}$$

where

$$(BR^i)^\epsilon(s) = \operatorname{argmax}\{\Pi^t(s/\alpha^t) : \alpha^t \in \Sigma^\epsilon, t \in T_i\}$$

and

$$\Sigma^\epsilon = \bigcap_{i \in T} (\Sigma^i)^\epsilon.$$

Hence by the maximum principle  $(BR^i)^\epsilon$  is compact valued and upper-semicontinuous.

STEP 1.  $\Gamma^\epsilon$  has at least one T.S.N.E.

The correspondence,

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<sup>4</sup>The arguments used here are as in Dubey-Geanakoplos [1, 2].

$$BR^\epsilon(s) = \prod_{i \in T} (BR^i)^i : \sum^\epsilon \rightarrow \sum^\epsilon$$

where  $s$  is a type symmetric play, satisfies all the conditions of Kakutani fixed point theorem and therefore admits a fixed point  $BR^\epsilon(\hat{s}^\epsilon) \ni \hat{s}^\epsilon$ . It is easily verified that  $\hat{s}^\epsilon$  is a T.S.N.E. Thereafter  $\hat{s}^\epsilon, \hat{p}^\epsilon$  indicates a T.S.N.E.

STEP 2.  $\rho \geq 0$  at any T.S.N.E.

If  $\rho < 0$  then  $D^t < 0 \quad \forall t \in T_i$  and some trader type  $i$ . Then  $t$  could bid for more of a commodity without ending insolvent and increasing his utility by monotonicity. Therefore a contradiction.

STEP 3. There exists  $c > 0$  such that  $\forall \epsilon$  sufficiently small  $\hat{p}^\epsilon > ce$ , where  $e = (1, \dots, 1)$  of suitable dimensionality.

Suppose  $\lim_{\epsilon \rightarrow 0} p_j^\epsilon \rightarrow 0$  for some  $j$ . Recall that at equilibrium,

$$p_j = \frac{\int_T b_j^t w^t + \epsilon}{(\int_T q_j^\epsilon + \epsilon) p_{m+1}}$$

where  $p_{m+1} = 1$ . If for a commodity  $\ell \neq j, p_\ell^\epsilon \rightarrow 0$  then trader  $t$  could have made a bid smaller by  $\Delta$  or otherwise if all  $p_j^\epsilon \rightarrow 0 \quad \forall j = 1, \dots, m$ , then there definitely exists a trader of type  $i$  with  $\Delta > 0$  of banknotes. The argument also goes through the gold market where  $p_{m+1} = 1$ . Thus, this type  $i$  could bid  $\Delta$  amount of money to obtain  $\Delta/p_j^\epsilon \rightarrow \infty$  of  $j$  since  $p_j^\epsilon \rightarrow 0$ . Let

$$\left\{ \frac{\partial \phi^t}{\partial x_j^t}(y) : y \leq \bar{G} \right\}$$

where  $\bar{G} = k \sum_{i=1}^n a_{m+1}^i$ .

Since nobody can acquire more than  $\bar{G}$  at equilibrium we see that because of strict monotonicity of utilities, the net gain in utility for  $t \in T_i$  as a result of his action is at least

$$\left\{ \frac{\partial \phi^t / \partial x_j^t(y)}{p_j^\epsilon} - 1 \right\} \Delta.$$

for a  $\Delta$  sufficiently small. This quantity in parentheses has to be non-positive since we are at equilibrium. So,

$$\frac{\partial \phi^t}{\partial x_j^t}(y) < p_j^\epsilon.$$

We will construct the T.S.N.E.  $\hat{s}$  and the corresponding vector of prices  $\hat{p}$  of  $\Gamma$  as a limit of  $\Gamma^\epsilon$  and its equilibrium. So, we take a sequence of  $\epsilon$ 's which defines a limiting  $\hat{s}$ .

STEP 4. Select a sequence of  $\epsilon$  and subsequences of subsequences so that:

- (i)  $\lim_{\epsilon \rightarrow 0} \hat{s}^\epsilon = s$ . This is possible since strategy spaces are *bounded*.
- (ii) Choose a subsequence so that  $E^\epsilon = \sum_{j=1}^m p_j^\epsilon$  is convergent to either  $\psi < \infty$  or  $\infty$ .
- (iii)  $p_j^\epsilon / \sum_{j=1}^m p_j^\epsilon$  converges  $\forall j = 1, \dots, m$ .
  - if  $E^\epsilon < \infty$  define  $\lim_{\epsilon \rightarrow 0} p_j^\epsilon = p_j$  and  $p_{m+1} = 1$ ,
  - if  $E^\epsilon = \infty$  define  $\lim_{\epsilon \rightarrow 0} [(p_j/E)^\epsilon] = p_j$  and  $p_{m+1} = 0$ .

STEP 5. If  $E^\epsilon \rightarrow \infty$

- (i)  $p_j^\epsilon \rightarrow \infty, \forall j = 1, \dots, m$ .

Clearly  $p_j^\epsilon \rightarrow \infty$  for some  $j$ . Let  $p_\ell^\epsilon$  be bounded for some  $\ell \neq j$ . Take player  $t \in T_i$  who has a positive amount of  $j$ . He can sell  $\Delta$  to obtain an enormous amount of banknotes  $B$  so as to bid and acquire  $B/p_\ell^\epsilon >> 0$ . So he has a *positive* net gain in utility. So, we get a contradiction that we were at a N.E. In addition, bankruptcy could be problematically prevented since  $B > k \sum_{i=1}^n u^i$ .

- (ii)  $q_j^\epsilon \rightarrow 0$  and  $\frac{(b_j^t)^\epsilon w^t}{p_j^\epsilon} \epsilon \rightarrow 0 \quad \forall i = 1, \dots, n$

$$p_j^\epsilon \leq \frac{k \sum_{i=1}^n a_{m+1}^i + \epsilon}{\sum_{i=1}^n (q_j^i)^\epsilon + \epsilon} \quad \text{and} \quad \sum_{j=1}^{m+1} (b_j^i)^\epsilon w^i \leq k \sum_{i=1}^n a_{m+1}^i.$$

Thus, by (i)  $(q_j^i)^\epsilon \rightarrow 0$  and  $\frac{(b_j^t)^\epsilon w^t}{p_j^\epsilon} \rightarrow 0$ .

STEP 6. If  $E^\epsilon < \infty$  and  $p_{m+1} = 1$  then by Step 3  $p^\epsilon > 0 \quad \forall j = 1, \dots, m$ . Moreover  $\hat{s}$  is a T.S.N.E. It follows from Step 4 that  $\lim_{\epsilon \rightarrow 0} (BR^i)^\epsilon = BR^i$ . From Step 5 we know that  $\hat{s}_i = \lim_{\epsilon \rightarrow 0} (s^i)^\epsilon \quad \forall i$  except when  $p_{m+1} = 0$ . So,  $\hat{s}$  maximizes  $\phi^i$  on  $\Gamma$  and therefore  $\hat{s}$  is a T.S.N.E.

STEP 7. Any limit  $s^\epsilon$  of T.S.N.E. for  $\Gamma^\epsilon$  is an active T.S.N.E.

By Step 2,  $\rho > -1$  and by Steps 5 and 6,  $p_j > 0$ .

Q.E.D.

**THEOREM 2.** For a sufficiently high gearing ratio  $k$  and a sufficiently harsh penalty  $\mu$  which entails no bankruptcy, the strategic market game  $\Gamma(n, \phi^i, a^i, s^i)$  will produce  $\rho = 0$  and give the same relative prices and distribution as the Walrasian equilibrium of the associated exchange economy  $E(n, \phi^i, a^i)$ .

**PROOF**

STEP 1.  $\rho \rightarrow 0$  as  $k \rightarrow \infty$  at any T.S.N.E.

By Step 2 of Theorem 1,  $\rho \geq 0$ . Now let  $\rho > 0$ . From (4),

$$(*) \quad \int_T v^i > k \int_T u^i > 0.$$

Since  $p_{m+1} = 1$  and  $a_{m+1}^i < \infty \forall i = 1, \dots, n$ , pick  $\bar{a}^i = \max[a_{m+1}^1, \dots, a_{m+1}^n]$  and  $k$  such that,

$$(**) \quad \frac{k \int_T u^i}{n} > p_{m+1} \bar{a}^i.$$

This is possible by the upper bound on  $u$ 's by (2) since  $k$  is an unbounded exogenous parameter. By (\*), (\*\*), for agent  $i$   $D^i > 0$ . Similarly, for the other players. So, contradiction. Thus, there exists  $k$  high enough for which  $\int_T u^i = k \int_T v^i$ .

STEP 2.  $(x, p)$  arising from the optimal play  $\hat{s}$  is a W.E.

By the no bankruptcy condition and Step 1,  $px^i \geq pa^i \forall i = 1, \dots, n$ . So, by utility maximization and local non-satiation  $\sum_{i=1}^n x^i = \sum_{i=1}^n a^i$  which implies  $px^i = pa^i \forall i = 1, \dots, n$ . Q.E.D.

**THEOREM 3.** *The strategic market game  $\Gamma(n, \phi^i, a^i, s^i)$  with a continuum of traders of  $n$  types and a mutual bank for any non-zero amount of gold and any gearing ratio  $k \geq 1$  and bankruptcy penalty  $\mu$  has an active T.S.N.E.*

**PROOF**

STEP 0

(i) Now since there exists the potentiality of bankruptcy the effective payoff function is

$$\Pi^i(s) = \phi^i(x^i(s)) - \mu \max[0, D^i].$$

(ii) From Step 0 of Theorem 1, we know that  $\phi^i(x^i(s))$  is concave. Also, 0 is a trivially concave. It is easily seen that  $D^i$  is concave (linear) as well. So,  $\Pi^i(s)$  is concave and continuous. Hence

$$(BR^i)^\epsilon : \sum^\epsilon \rightarrow (\sum^\epsilon)^\epsilon \text{ is convex.}$$

So, by the maximum principle  $(BR^i)^\epsilon$  is compact valued and upper-semicontinuous.

The rest of the formulation is as in Step 0 of Theorem 1.

STEP 1.  $I^\epsilon$  has at least one T.S.N.E.

As in Step 1 of Theorem 1.

STEP 2.  $\rho \geq 0$  at any T.S.N.E.

As in Step 2 of Theorem 2.

STEP 3. *There exists  $c > 0$  such that  $\forall \epsilon > 0$  sufficiently small  $\hat{p}^\epsilon > ce$ , where  $e = (1, \dots, 1)$  of suitable dimensionality.*

Suppose  $\hat{p}^\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$  for  $j$ . Then choose player  $t$ . He could borrow  $\Delta$  of banknotes to buy  $\Delta/p_j^\epsilon \rightarrow \infty$  since  $p_j^\epsilon \rightarrow 0$ . This transaction would incur him a bankruptcy penalty  $\mu\Delta < \infty$  since both  $\Delta$  and  $\mu$  are bounded from the above. Now let,

$$\left\{ \frac{\partial \phi^t}{\partial x_j^t}(y) : y \leq \bar{G} \right\}.$$

Since nobody can get more than  $\bar{G}$  at equilibrium, we see that because of strict monotonicity of preferences, the net gain in utility of trader  $t$  because of his action is at least,

$$\left\{ \frac{\partial \phi^t / \partial x_j^t(y)}{p_j^\epsilon} - \mu \rho \right\} \Delta,$$

for a  $\Delta$  sufficiently small. Note that by Step 2,  $\rho \geq 0$ . In addition  $\rho$  is bounded from the above. Since  $k f_T u^t(1+\rho)$  has to be repayed at the end of trade to the mutual bank and consequently some trader  $t$  has to pay a bankruptcy penalty equal to at least  $\mu \left[ \frac{k f_T u^t \rho}{n} \right]$ . Therefore, if  $\rho \rightarrow \infty$  then  $i$  could have been less worse off than he is now at our assumed equilibrium either by bidding less or if he had deposited less gold at the mutual bank and therefore deriving excess utility from gold directly as a commodity at the end.

The expression in the brackets has to be non-positive,

$$p_j^\epsilon \geq \frac{\partial \phi^t / \partial x_j^t(y)}{\mu \rho}.$$

We will construct again the T.S.N.E. and the corresponding  $\rho$  of  $\Gamma$  as a limit of  $\Gamma^\epsilon$  and its associated equilibrium. For this purpose take a sequence  $\epsilon$  which defines a limiting pair  $\delta$ .

STEP 4. (i)-(iii) as of Step 4 of Theorem 1.

STEP 5. *If  $E^\epsilon \rightarrow \infty$  then*

(i)  $p_j^\epsilon \rightarrow \infty$  for some  $\forall j = 1, \dots, m$ .

Clearly  $p_j^\epsilon \rightarrow \infty$  for some  $j$ . Let  $p_\ell^\epsilon$  be bounded for some  $\ell \neq j$ . Consider a player who has a positive amount of  $j$ . Then this player is in a position to borrow a small amount  $\Delta$  of banknotes and offers for sale  $(1+\rho)\Delta/p_j$  of commodity  $j$  and afterwards buys  $\Delta(b_\ell^t)^\epsilon w^t/p_\ell^\epsilon$  where in this case  $(b_\ell^t)^\epsilon = 1$ . Consequently he

can return  $(1+\rho)\Delta$  banknotes to the mutual bank. Therefore, the net gain in utility is positive and therefore we get a contradiction.

(ii)  $(q_j^t)^\epsilon \rightarrow 0$  and  $(b_j^t)^\epsilon w^t / p_j^\epsilon \rightarrow 0 \quad \forall t \in T_i$  and  $i = 1, \dots, n$

$$p_j^\epsilon \leq \frac{k \sum_{i=1}^n a_{m+1}^i + \epsilon}{\sum_{i=1}^n (q_j^i)^\epsilon + \epsilon} \quad \text{and} \quad \sum_{j=1}^{m+1} (b_j^i)^\epsilon w^t \leq k \sum_{i=1}^n a_{m+1}^i.$$

So by (i),  $(q_j^t)^\epsilon \rightarrow 0$  and  $(b_j^t)^\epsilon w^t / p_j^\epsilon \rightarrow 0$ .

STEP 6. If  $E^\epsilon < \infty$  and  $p_{m+1} = 1$  then by Step 3  $p^\epsilon > 0 \quad \forall j = 1, \dots, m$ . In addition  $\hat{s}$  is a T.S.N.E. Same argument as in Step 6 of Theorem 1.

STEP 7. Any limit  $s^\epsilon$  of T.N.S.E. for  $T^\epsilon$  is an active T.N.S.E.

As in Step 7 of Theorem 1.

Q.E.D.

**THEOREM 4.** For a sufficiently high gearing ratio  $k$  and a sufficiently harsh default penalty  $\mu$ , the strategic market game  $I(n, \phi^i, a^i, s^i)$  will have T.S.N.E.'s which do not involve bankruptcy, produce an endogenous interest rate  $\rho = 0$  and give the same relative prices and distribution as the Walrasian equilibrium of the associated exchange economy  $E(n, \phi^i, a^i)$ .

**PROOF.** It only suffices that for a high enough  $k$  at any T.S.N.E.  $\rho \rightarrow 0$  and furthermore that no agent bankrupts. Then we are back at the argument of Theorem 2.

STEP 1. No agent bankrupts when  $k \rightarrow \infty$  if  $E^\epsilon = \infty$ .

Recall that  $D^t$  is the debt of a player  $t$ . If  $D^t < 0$  then player  $t$  could have always borrowed  $\Delta$  less amount of banknotes and not bid them for a commodity  $j$  whose  $(p_j^\epsilon) \rightarrow \infty$  and finally obtain,

$$(x_j^t)^\epsilon = (x_j^t)^\epsilon - \frac{\Delta}{p_j^\epsilon}.$$

Therefore, he is saving the bankruptcy penalty  $(1+\rho)\Delta\mu$  without reducing his utility for a sufficiently small  $\Delta$  as  $p_j^\epsilon \rightarrow \infty$ .

STEP 2.  $\rho \rightarrow 0$  as  $k \rightarrow \infty$  at any T.S.N.E. Also, no agent bankrupts in general.

Let  $\rho > 0$  along a subsequence of  $\epsilon$ . Then consider a type-symmetric play. Then for some player  $t$ ,  $D^t > \rho k f_T u^t / n$  if bankruptcy has occurred. If not then go back to Step 1 of Theorem 2. Now for the case of bankruptcy, if

$$- \frac{\mu \rho k \int_T \mu^t}{n} + \phi^t(\sum_i a^i) < \min_{1 \leq t' \leq n} \phi^{t'}(0) - \frac{m+1}{M}$$

for sufficiently high  $k$ , then each  $t \in T_i$  could have improved upon this allocation by borrowing  $m+1/M$  less banknotes and selecting  $s' = \{ \dots, 1/M, \dots, 1/M; \dots \}$ . Therefore, we arrived at a contradiction.

Finally, note that easily one can show the converse of the theorem along the same lines of Dubey-Geanakoplos [1]. Q.E.D.

### 2.3. An Example

In a previous paper an example of a loan market for gold was given. We extend this example here to a bank with fractional gold reserves. The initial endowment of traders of type 1 is assumed to be  $(2, 0, 0)$  and that of traders of type 2 is  $(0, 2, A)$  where the third commodity is gold and can be used as money. A strategy by a trader of type 1 is of the form  $(v^1, b_j^1, q_j^1)$  where  $j = 1, 2$  and  $v$  is the amount of I.O.U. notes bid to obtain banknotes. A strategy by a trader of type 2 is of the form  $(u, v^2, b_j^2, q_j^2)$  where  $j = 1, 2$   $u$  is the amount of gold deposited in the mutual bank and  $v$  is the amount of I.O.U. notes bid to obtain banknotes.

In this example because of the special structure and the assumption that all traders are of insignificant size we can simplify the notation even further. We assume that the price of money is set at  $p_3 = 1$ . The other prices are denoted by  $p_1$  and  $p_2$ . Traders of the first type only will borrow and those of the second type only will form the mutual bank and then borrow from it. Because of the initial endowments only those of the first type can sell the first good and only of those of the second type can see the second good.

We assume that there is a type symmetric noncooperative equilibrium and that setting  $\mu = 10$  provides a sufficiently harsh penalty to avoid strategic bankruptcy. We furthermore assume that traders borrow only to spend on buying goods (i.e. they do not borrow to hoard). Given these simplifications we can write the payoff functions as follows:<sup>5</sup>

$$(1) \quad \Pi_1 = \sqrt{(2 - q_1^1)(v^1/(1+\rho)p_2)} + (p_1 q_1^1 - v^1)^+ + \mu (p_1 q_1^1 - v^1)^-$$

$$(2) \quad \Pi_2 = \sqrt{(2 - q_1^1)(v^2/(1+\rho)p_1)} + (p_2 q_2^2 - v^2 + k\rho A)^+ + \mu (p_2 q_2^2 - v^2 + k\rho A)^-$$

where  $p_1 = v^2/(1+\rho)q_1^1$  and  $p_2 = u/(1+\rho)q_2^2$ . also  $1+\rho = (v^1 + v^2)/ku$  and  $v^1 = p_1 q_1^1$  and  $v^2 = p_2 q_2^2$ .

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<sup>5</sup>The notation  $( )^+$  or  $( )^-$  stands for the positive or negative part of the item in the brackets, or zero.



Treating  $p_1$  and  $p_2$  as parameters and making the observation that if  $k > 1$  and the rate of interest is greater than zero all gold will go into the bank, the first order optimization conditions give:

$$(11) \quad \frac{1}{2} \sqrt{v^1 / (1+\rho) p_2 (2 - q_1^1)} = p_1 \quad \text{and}$$

$$(12) \quad \frac{1}{2} \sqrt{(2 - q_1^1) / (1+\rho) p_2 v^1} = 1$$

$$(13) \quad \frac{1}{2} \sqrt{v^2 / (1+\rho) p_1 (2 - q_2^2)} = p_2 \quad \text{and}$$

$$(14) \quad \frac{1}{2} \sqrt{(2 - q_2^2) / (1+\rho) p_1 v^2} = 1.$$

We can check that for  $\mathcal{A} = .5$  the following values shown in Table 1.

TABLE 1

k	$\rho$	$q_1^1$	$q_2^2$	$v^2$	$v^1$	$p_1$	$p_2$
4	0	1	1	1	1	1	1
2	0	1	1	.5	.5	.5	.5
1.8	.1096	1	.8967	.4734	.4266	.4733	.4759
1	.9091	1	.4306	.3280	.1719	.3281	.3992

### 3. SENSITIVITY ANALYSIS

#### 3.1. Inflation and the Money Supply

In Table 2 the first entry has  $k = 4$ , thus the supply of banknotes is  $M = 2$ . At equilibrium the rate of interest cannot be negative if loans are available because an individual could arbitrage by borrowing, thus the bids will be  $v^1 = v^2 = 1$ . However price will no longer be uniquely determined. Any set of prices  $.5 \leq p_1 = p_2 \leq 1$  is in equilibrium. If prices for the two goods are between .5 and 1 individuals hoard the extra banknotes. If we assume that gold although used for reserves is not a means of payment we can have a market for gold in the hands of the public, its price in terms of banknotes is twice that of the two other goods. In essence it has two prices because of the strategic restrictions on it. Its price in trade is determined in the open market and has to bear the appropriate marginal relationship to other goods in competitive trade, but its price in settling loans at the bank is one to one with banknotes.

There is yet another solution here where  $u = .25$ , thus  $M = 1$  and the price level is  $p_1 = p_2 = .5$ . If the gold remaining in circulation is counted as part of the money supply, then it can be used instead of banknotes

in exchange. However it will be either hoarded or traded only in an extra money market between gold and banknotes where its price will be  $k$  in terms of banknotes.

Legally the total money supply is 1.25 of which  $4/5$  are banknotes and  $1/5$  gold. But gold in this model will be no more used as a direct means of payment than \$20 United States gold coins are used as a means of payment at the local grocery. In the United States, as recently as the 1950s it was not uncommon to be given silver dollars in payment; they are still legal tender and can be used by those who choose to pay with them, but a new market exists between them and paper money and although as a *legal* means of payment they trade at par with a federal reserve note, as a consumer (or potential producer) good they trade at a premium.

### 3.2. The Type and Intensity of the Default Penalty

If exchange is modeled as a game of strategy in which credit is granted. In general it will be feasible for the system to reach a state in which an individual is unable to honor his commitments. It is a logical necessity and not merely an institutional convenience that a default penalty be specified in order to complete the rules of the game so that payoffs can be computed in the case of default.

Penalties can be highly economic (such as the confiscation and sale of goods) or societal such as exile, prison or loss of civil rights. The details are highly institutional. If, for example we wish to consider garnishing or confiscation of property we must specify what can be removed and how it is to be used to extinguish the loan. For our purposes here there are two items that matter in the one period model. They are that the penalty serves as a disincentive to the borrower to default and that in the case of default the penalty is such that the loss to society as a whole is in some sense minimized.

In an economy without exogenous uncertainty and enough money<sup>6</sup> the threat of bankruptcy or default is only strategic, thus if a penalty is harsh enough to prevent bankruptcy it is efficient. But in an economy with exogenous uncertainty the bankruptcy penalty is in essence a public good reflecting society's joint willingness to risk default.<sup>7</sup>

As there is no exogenous uncertainty in the model above, our remarks can be confined to three levels of the penalty. They are when the penalty is so weak that default pays; when it is just strong enough to discourage strategic default and when it is more than strong enough to discourage default. These three levels

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<sup>6</sup>With the money well distributed or loaned at zero interest.

<sup>7</sup>The nature of the sharing of the damage caused by a default is critical (see Dubey, Geanakoplos and Shubik [1988]).

of  $\mu$  should be considered with three levels of  $k$ ; not enough to provide a sufficient money supply; just enough to provide a sufficient money supply and more than enough to do so. There are nine cases illustrated in Table 2. These conditions can be stated precisely.

We assume that at the minimal gold transaction C.E.:

$$\sum_{i=1}^n \sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] \geq \sum_{i=1}^n a_{m+1}^i.$$

The gearing ratio  $k$  will be just sufficient if:

$$\sum_{i=1}^n \sum_{j=1}^m p_{jk} \max[0, (x_{jk}^i - a_j^i)] = k \sum_{i=1}^n a_{m+1}^i.$$

We call this  $k$ ,  $k^*$ .

If the amount of money is  $M = k^* a_{m+1}$  there will be a default penalty such that at  $\mu = \mu^*$  no one defaults at the price level which uses all of  $M$  in transactions.

TABLE 2

	$k < k^*$	$k = k^*$	$k > k^*$
$\mu < \mu^*$	1 Strategic default/Deflation An N.E. with default dead weight loss of penalty  $\rho > 0$	2 Strategic Default An N.E. with default dead weight loss of penalty  $\rho > 0$	3 Possibly strategic default Deflation or inflation An N.E. with default dead weight loss--otherwise one or more C.E.s as N.E.s with a continuum of deflated or inflated prices  $\rho \geq 0$
$\mu = \mu^*$	4 Strategic default/Deflation A transaction constrained N.E. with default and a unique price level  $\rho > 0$	5 No default A unique C.E. as an N.E. with a unique price and transactions summing to $M$  $\rho = 0$	6 No default/Inflation One or more C.E.s as N.E.s with a continuum of prices bounded above by transactions summing to $M$  $\rho = 0$
$\mu > \mu^*$	7 Possibly Strategic Default Deflation A transactions constrained N.E. with possible default and a continuum of deflated prices  $\rho \geq 0$	8 No default/Deflation One or more C.E.s as N.E.s with upper bound price level as case 5 but a continuum of deflated prices  $\rho = 0$	9 No default/Deflation or Inflation One or more C.E.s as N.E.s and continua of deflated and inflated prices  $\rho = 0$

**REMARKS**

- (i) In all cases the upper bound on prices determined by  $k$  is greater than the lower bound on prices determined by  $\mu$ ; otherwise equilibrium price vectors do *not* exist.
- (ii)
  - if  $\mu/k = \mu^*/k^*$  then there exists a *unique* C.E.
  - if  $\mu/k > \mu^*/k^*$  then we have *deflation*.
  - if  $\mu/k < \mu^*/k^*$  then we have *inflation*.

## APPENDIX I

### Type I: No bankruptcy

Case (a): Loans paid back by banknotes

Case (b): Loans paid back by final gold holdings

Case (c): Loans paid back by banknotes and final gold holdings

### Type II: Bankruptcy

Case (a): Loans paid back *partially* by banknotes

Case (b): Loans paid back *partially* by final gold holdings

Case (c): Loans paid back *partially* by banknotes and final gold holdings

### Example: I(a)

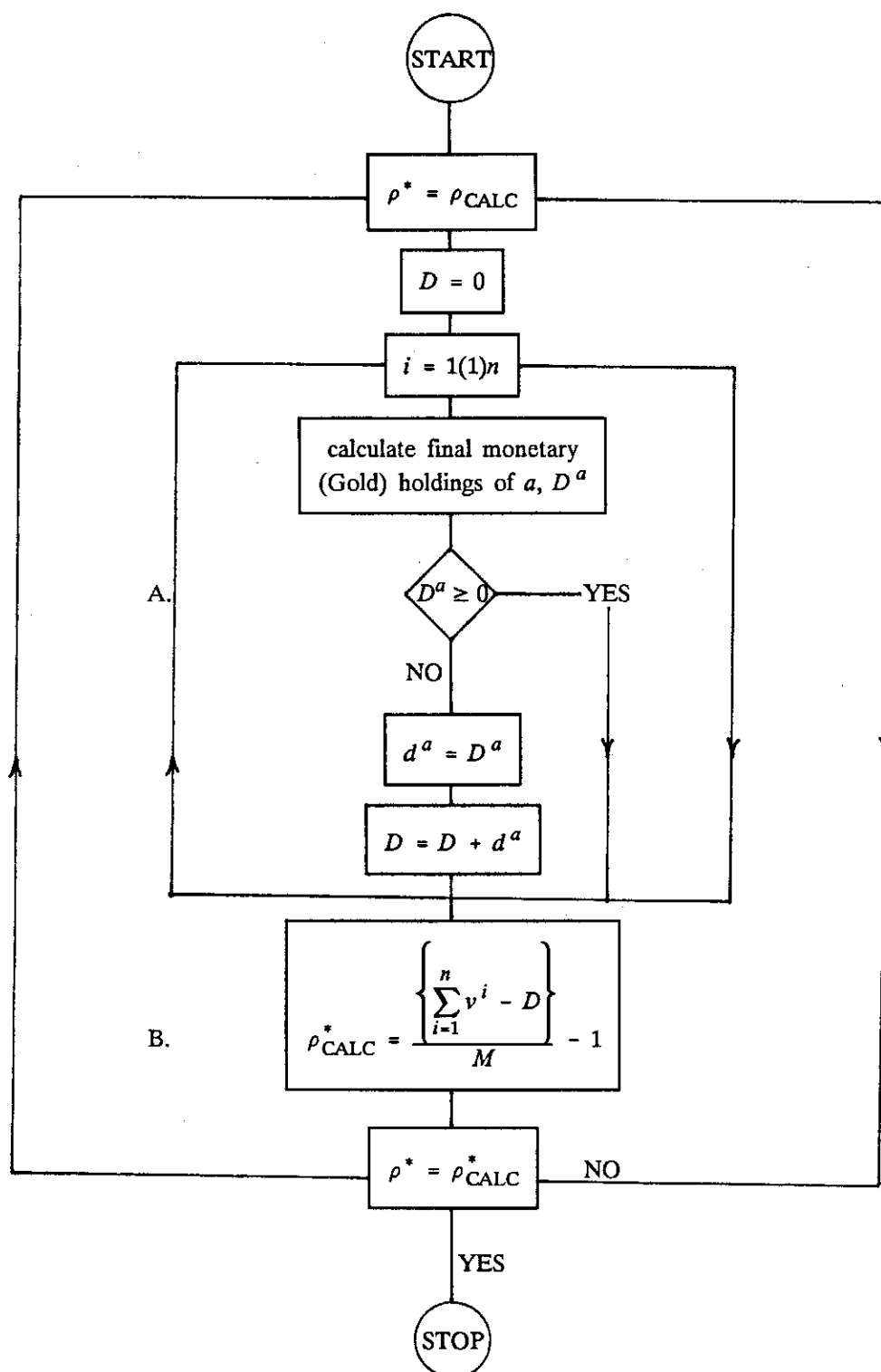
$$D^i = \left\{ a_{m+1}^i - u^i \right\} + \left\{ \frac{v^i}{1+\rho} - v^i \right\} + \left\{ \sum_{j=1}^{n+1} p_j q_j^i - \sum_{j=1}^{m+1} b_j^i w^i \right\} \\ + \theta^i \left\{ M - \frac{\sum_{i=1}^n v^i}{1+\rho} + \frac{\sum_{i=1}^n v^i (1+\rho)}{1+\rho} \right\}$$

After rearranging and substituting the last term,

$$D^i = \left\{ a_{m+1}^i - u^i \right\} + \left\{ \frac{v^i}{1+\rho} - v^i \right\} + \left\{ \sum_{j=1}^{n+1} p_j q_j^i - \sum_{j=1}^{m+1} b_j^i w^i \right\} + k u^i (1+\rho)$$

Note that since there does *not* exist bankruptcy,  $\rho = \rho^*$ ,  $\hat{v}^i = v^i$ ,  $g_L^i = 0$ .

## APPENDIX II



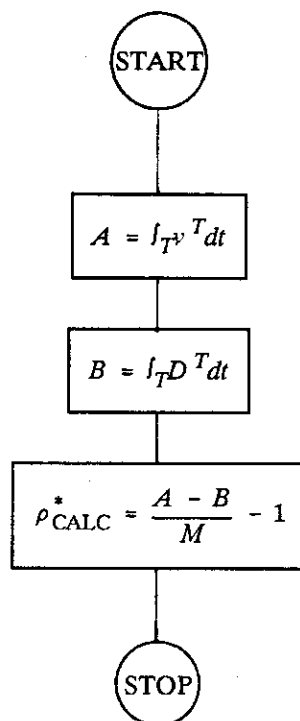
NOTE: In the first iteration we set  $\rho^* = \rho_{\text{EX ANTE}}$ .

CALCULATION OF THE EX POST INTEREST RATE  
[DISCRETE CASE]

## APPENDIX III

The inner loop  $\mathcal{A}$  and the iterative formation of  $\rho^*$  have to be replaced by the following routine.

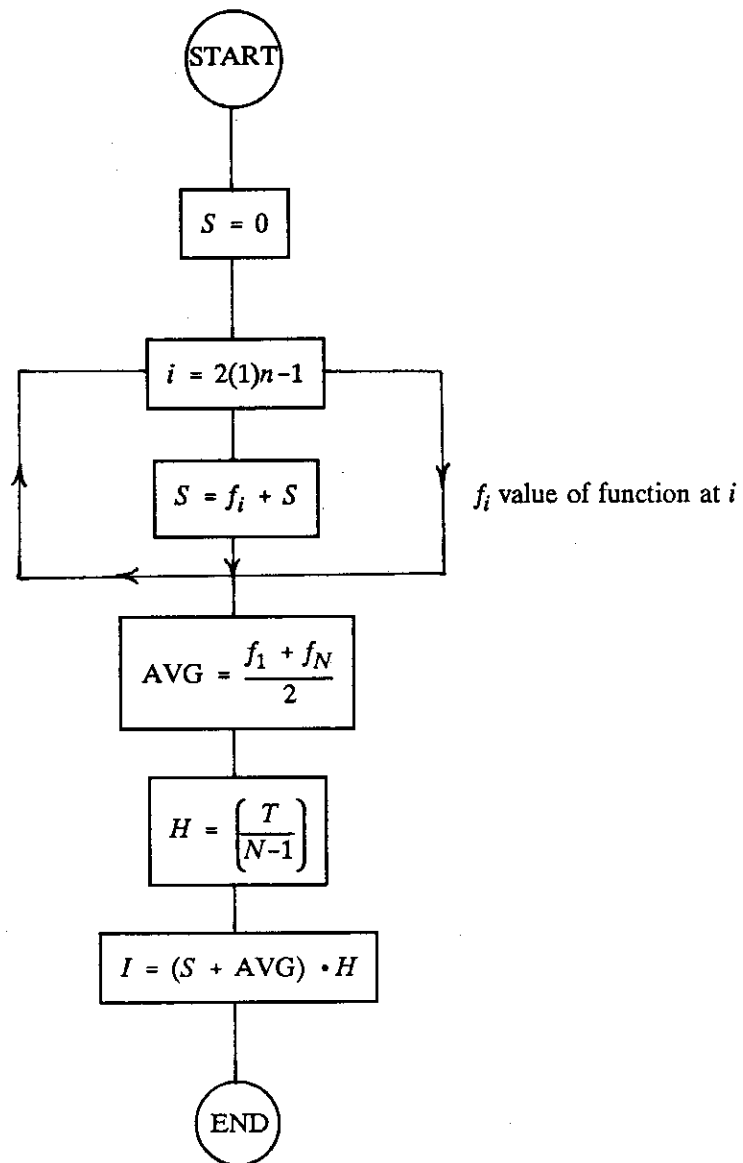
III.A



We now need a routine simple enough for the evaluation of the integral.  
Let for example

$$I = \int_T f(t) dt$$

III.B



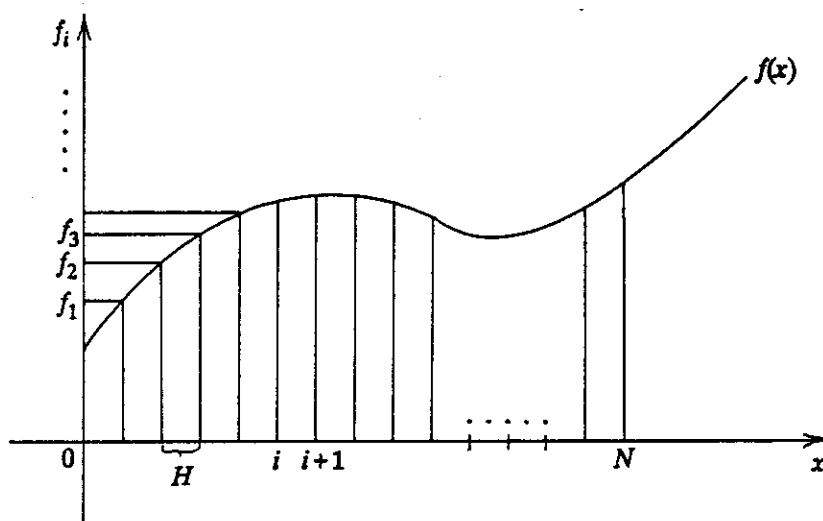
where,  $S = \sum_{i=1}^{N-1} f_i$ ,  $AVG = \frac{f_1 + f_N}{2}$

CALCULATION OF THE EX POST INTEREST RATE  
[CONTINUUM CASE]



LEMMA:<sup>1</sup>  $I = (S + \text{AVG}) \cdot H$

PROOF:



$\therefore$  Area of the sum of the trapezoids gives us the integral

$$\begin{aligned}
 I &= \sum_{i=1}^{N-1} \left( \frac{f_i + f_{i+1}}{2} \right) H \\
 &= \frac{H}{2} \left[ f_1 + 2 \sum_{i=1}^{N-1} f_i + f_N \right] \\
 &= H \left[ \underbrace{\frac{g_1 + g_N}{2}}_{\text{AVG}} + \underbrace{\sum_{i=1}^{N-1} g_i}_{S} \right]
 \end{aligned}$$

$$\therefore I = (S + \text{AVG}) \cdot H.$$

Q.E.D.

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<sup>1</sup>More sophisticated algorithms for the evaluation of integrals can be constructed. One such algorithm is the Romberg Algorithm (see R. L. Burden and J. Douglas Faires, "Numerical Analysis," pp. 177-182).

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